1. (a) If f is integrable and bounded by M on [a, b] then show that

$$\left|\int_{a}^{b} f(x)dx\right| \le (b-a)M.$$

Hint: You may want to use the triangle inequality,

$$\left|\int_{a}^{b} f(x)dx\right| \le \int_{a}^{b} |f(x)|dx$$

- (b) Let f be integrable on [a, b], show that $A(x) = \int_a^x f(x) dx$ is continuous for $x \in [a, b]$.
- (c) **(Bonus):** Prove the triangle inequality.

Remark: This is an example of how the integral can be used to make "bad" functions into nice ones. Even if f has discontinuities, the integral will still be continuous. If you have a nice (continuous) function to begin with it makes it even nicer (differentiable) by FTC!

2. (a) Suppose that $f \ge 0$ is continuous integrable on [a, b] and

$$\int_{a}^{b} f(x) = 0.$$

Prove that f(x) = 0 on [a, b].

- (b) Is it still true if f is not always positive?
- (c) Is it still true if f is not continuous?

Remark: This is one way to determine global properties of your functions by just knowing information about the integral. This becomes very useful in practice since suppose we want to show that f(x) = g(x) but we are not given enough information to do it algebraically. What this questions shows is that we can instead show

$$\int_{a}^{b} |f(x) - g(x)| dx = 0 \quad \text{or} \quad \int_{a}^{b} (f(x) - g(x))^{2} dx = 0$$

Which can be a lot easier!

3. Let f, g be integrable on [a, b]. By this point in the course (I hope) you have noticed that in general

$$\int_{a}^{b} fgdx \neq \int_{a}^{b} fdx \int_{a}^{b} gdx.$$

So do we just give up? Clearly the answer is no, or else I wouldn't be typing this. The goal of this problem is to salvage the above, we should be able to determine some information about the integral of fg with only information about f and g. We will prove a very important estimate in calculus, called the **Cauchy-Schwarz** inequality,

$$\int_{a}^{b} fgdx \le \sqrt{\int_{a}^{b} f^{2}dx \int_{a}^{b} g^{2}dx}$$

(a) Prove that for all $t \in \mathbb{R}$, that

$$t^2 \int_a^b f^2 dx + 2t \int_a^b fg dx + \int_a^b g^2 dx \ge 0$$

Hint: Examine the integral of $(tf + g)^2$.

(b) If $\int_a^b f^2 dx \neq 0$, show that this quadratic is minimized when

$$t = -\frac{\int_a^b fgdx}{\int_a^b f^2dx}.$$

- (c) Deduce the Cauchy-Schwarz inequality.
- (d) Use the Cauchy-Schwarz inequality to prove that

$$\int_0^\pi x^n \sin x \, dx \le \frac{\pi^{n+1}}{\sqrt{4n+2}}$$

1. (a) By the triangle inequality, we have that

$$\left|\int_{a}^{b} f(x)dx\right| \leq \int_{a}^{b} |f(x)|dx \leq \int_{a}^{b} Mdx = M(b-a)$$

(b) Let us look at A(x) - A(y), without loss of generality assume that y < x. So

$$|A(x) - A(y)| = \left| \int_{a}^{x} f(t)dt - \int_{a}^{y} f(t)dt \right|$$
$$= \left| \int_{y}^{x} f(t)dt \right|$$
$$\leq M(x - y)$$
$$= M|x - y|$$

So if we let $\varepsilon > 0$, and $\delta = \frac{\varepsilon}{M}$, then if $|x - y| < \delta$ we will have

$$|A(x) - A(y)| < \varepsilon.$$

Implying that A is continuous.

- (c) If you give me a full written proof before the midterm, I will give a few marks back on your worst assignment. Note that you need to prove that $\int_a^b |f| dx$ exists!
- 2. (a) Let us suppose that there is an $x_0 \in [a, b]$ such that $f(x_0) > 0$. So by continuity there is a $\delta > 0$ such that $|x x_0| < \delta$ implies that $|f(x) f(x_0)| < f(x_0)/2$. This implies for $x \in [x_0 \delta, x_0 + \delta]$, that $f(x) > f(x_0)/2$. And thus

$$\int_{a}^{b} f(x)dx \ge \int_{x_0-\delta}^{x_0+\delta} f(x)dx \ge \int_{x_0-\delta}^{x_0+\delta} \frac{f(x_0)}{2}dx = 2\delta \frac{f(x_0)}{2} = \delta f(x_0) > 0.$$

This contradicts the fact that the integral was 0. Thus f(x) = 0 for all $x \in [a, b]$.

(b) No, let $f(x) = \cos x$

$$\int_0^{\pi} \cos x \, dx = [\sin x]_0^{\pi} = 0.$$

How ever $\cos(x)$ is clearly not 0.

- (c) No, we have seen many examples in class. In particular f(1) = 1, and f(x) = 0 otherwise. f is integrable on [-1, 1] with integral 0.
- 3. (a) Let $t \in \mathbb{R}$. As suggested by the hint, we have

$$0 \leq \int_{a}^{b} (tf+g)dx$$

=
$$\int_{a}^{b} t^{2}f^{2} + 2tfg + g^{2}dx$$

=
$$t^{2} \int_{a}^{b} f^{2}dx + 2t \int_{a}^{b} fgdx + \int_{a}^{b} g^{2}dx$$

=
$$p(t)$$

(b) p(t) is a quadratic in t, so we can differentiate it to get,

$$p'(t) = 2t \int_a^b f^2 dx + 2 \int_a^b fg dx$$

By equating it to 0, we get the critical point is, at

$$t_c = -\frac{\int_a^b fgdx}{\int_a^b f^2dx}$$

Since $p''(t) = \int_a^b f^2 dx \ge 0$, we have that t_c is a minimum.

(c) If $\int_a^b f^2 dx = 0$, the inequality is trivially true. So if $\int_a^b f^2 dx > 0$, then

$$0 \leq p(t_c)$$

$$= \left(-\frac{\int_a^b fgdx}{\int_a^b f^2dx}\right)^2 \int_a^b f^2dx + 2\left(-\frac{\int_a^b fgdx}{\int_a^b f^2dx}\right) \int_a^b fgdx + \int_a^b g^2dx$$

$$= -\frac{\left(\int_a^b fgdx\right)}{\int_a^b f^2dx} + \int_a^b g^2dx$$

By rearranging we get the Cauchy-Schwarz inequality,

$$\left(\int_{a}^{b} fgdx\right)^{2} \leq \int_{a}^{b} f^{2}dx \int_{a}^{b} g^{2}dx.$$

(d) By the Cauchy-Schwarz inequality we have,

$$\int_0^\pi x^n \sin x dx \le \sqrt{\int_0^\pi x^{2n} dx \int_0^\pi \sin^2 x dx}$$

So let's do each integral on the right individually.

$$\int_0^{\pi} x^n dx = \left[\frac{x^{2n+1}}{2n+1}\right]_0^{\pi} = \frac{\pi^{2n+1}}{2n+1}$$
$$\int_0^{\pi} \sin^2 x dx = \int_0^{\pi} \frac{1}{2} - \frac{\cos(2x)}{2} dx = \left[\frac{x}{2} - \frac{\sin(2x)}{4}\right]_0^{\pi} = \frac{\pi}{2}$$

So we have

$$\int_0^{\pi} x^n \sin x \, dx \le \sqrt{\frac{\pi^{2n+1}}{2n+1} \frac{\pi}{2}} = \frac{\pi^{n+1}}{\sqrt{4n+2}}$$